

On non-stationary Lamé equation from WZW model and spin-1/2 XYZ chain

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ABSTRACT: We study the link between WZW model and the spin-1/2 XYZ chain. This is achieved by comparing the second-order differential equations from them. In the former case, the equation is the Ward-Takahashi identity satisfied by one-point toric conformal blocks. In the latter case, it arises from Baxter’s TQ relation. We find that the dimension of the representation space w.r.t. the V -valued primary field in these conformal blocks gets mapped to the total number of chain sites. By doing so, Stroganov’s “The Importance of being Odd” (cond-mat/0012035) can be consistently understood in terms of WZW model language. We first confirm this correspondence by taking a trigonometric limit of the XYZ chain. That eigenstates of the resultant two-body Sutherland model from Baxter’s TQ relation can be obtained by deforming toric conformal blocks supports our proposal.

1. Introduction

About twenty years ago, a series of pioneering papers [1, 2, 3] established an intriguing connection between XXX Gaudin and Wess-Zumino-Witten (WZW) models. That is, the problem of diagonalizing commuting Hamiltonians¹ of XXX Gaudin model is translated into solving Knizhnik-Zamolodchikov (KZ) equations defined on \mathbf{CP}^1 [6]. Indeed, Bethe roots of Bethe ansatz equations in the inhomogeneous XXX Gaudin model turn out to constitute solutions to KZ equations at critical level. Later on, the authors of [7, 8, 9] further extended this direction to the elliptic case. Certainly, their works are based on important investigations on both XYZ Gaudin model [10] and conformal field theory (CFT) on elliptic curves [11, 12, 13].

In this letter, we would like to add into the above picture a novel element: a relation between WZW model and the spin-1/2 XYZ chain as depicted in Fig. 1. By examining non-stationary Lamé equations on both sides we are able to interpret Stroganov's proposal (The Importance of being Odd) [14] from the viewpoint of CFT under the dictionary listed in Table 1.

Table 1: Dictionary		
	Spin-1/2 XYZ chain	WZW model
non-stationary Lamé eq.	Baxter's TQ eq.	KZB eq. (WT identity)
coupling const.	site number	dim. of \mathfrak{sl}_2 rep.
time	anisotropy parameter	torus moduli
space	spectral parameter	Cartan moduli

More precisely, in [15] Razumov and Stroganov made a conjecture about the exact ground-state eigenvalue of the transfer matrix in the spin-1/2 XYZ chain. This conjecture holds only for the odd chain site number and plays a crucial role in deriving the aforementioned Lamé equation [16]. On the other hand, one-point toric conformal blocks exist only when the dimension of the \mathfrak{sl}_2 representation space w.r.t. the inserted primary field is odd. It is thus tempting to connect these two facts through Table 1. As a test, we perform a trigonometric degeneration of the XYZ chain. Consequently, that eigenstates of Sutherland-type equations descending from Baxter's TQ relation reduce to Schur polynomials under certain limit is well reflected by imposing a corresponding constraint on WZW toric conformal blocks.

We organize this letter as follows. In the next section, we review how Lamé equations emerge from the spin-1/2 XYZ chain as a result of Baxter's TQ relation. We compare it with Knizhnik-Zamolodchikov-Bernard (KZB) equations in section 3. In section 4, we

¹Their simultaneous diagonalization is solved by algebraic Bethe ansatz [4] and Sklyanin's separation of variables [5]. Two approaches are essentially equivalent and amount to considering the quantized Gaudin spectral curve.

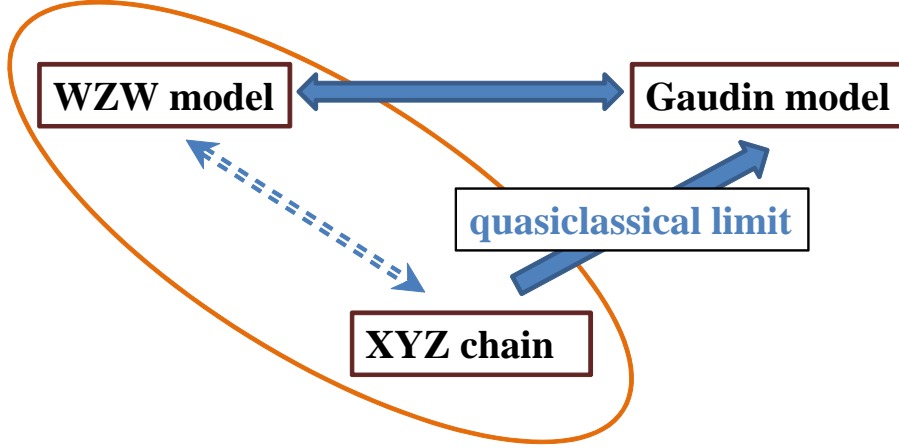


Figure 1: Two solid arrows above represent the known connections between spin-chain and WZW models. The encircled part indicates the novel relationship under consideration.

justify this comparison via a trigonometrical reduction. Finally, a summary is given in section 5.

2. Spin-1/2 XYZ chain side

Let us briefly review how the non-stationary Lamé equation is obtained from Baxter's TQ equation of the spin-1/2 XYZ chain [15, 16] whose Hamiltonian is described by

$$H_{XYZ} = \sum_{n=1}^M \left\{ J_X S_n^X S_{n+1}^X + J_Y S_n^Y S_{n+1}^Y + J_Z S_n^Z S_{n+1}^Z \right\}. \quad (2.1)$$

Here, $S_n^{X,Y,Z} = \sigma_n^{X,Y,Z}/2$ ($\sigma_n^{X,Y,Z}$: Pauli matrix) acts on the n -th site and the periodic boundary condition $S_{M+1}^{X,Y,Z} = S_1^{X,Y,Z}$ is imposed. Recall that the terminology XYZ means anisotropic J 's while the partial anisotropy $J_X = J_Y \neq J_Z$ (isotropy $J_X = J_Y = J_Z$) case is called the XXZ (XXX) chain. H_{XYZ} acts on the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_M$ where each V_n is a complex two-dimensional space \mathbf{C}^2 spanned by the up- and down-spin states.

A fundamental ingredient in integrable spin-chain models is the R matrix. For the spin-1/2 XYZ chain, its matrix elements are given by

$$R(z) = \begin{pmatrix} a(z) & 0 & 0 & d(z) \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ d(z) & 0 & 0 & a(z) \end{pmatrix}$$

where (nome: $q = e^{\pi i \tau}$)

$$\begin{aligned}
a(z) &= \rho \theta_4(2\eta|q) \theta_4(z|q) \theta_1(z + 2\eta|q), & b(z) &= \rho \theta_4(2\eta|q) \theta_1(z|q) \theta_4(z + 2\eta|q), \\
c(z) &= \rho \theta_1(2\eta|q) \theta_4(z|q) \theta_4(z + 2\eta|q), & d(z) &= \rho \theta_1(2\eta|q) \theta_1(z|q) \theta_1(z + 2\eta|q), \\
\rho &= \frac{2}{\theta_2(0|q^{\frac{1}{2}}) \theta_4(0|q)}, \\
\theta_1(z|q) &= -i \sum_{n \in \mathbf{Z}} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, & \theta_2(z|q) &= \sum_{n \in \mathbf{Z}} q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, \\
\theta_3(z|q) &= \sum_{n \in \mathbf{Z}} q^{n^2} e^{2n\pi i z}, & \theta_4(z|q) &= \sum_{n \in \mathbf{Z}} (-1)^n q^{n^2} e^{2n\pi i z}.
\end{aligned}$$

Note that (q, η) determines the anisotropy parameters of the XYZ chain through Jacobi's elliptic functions:

$$J_X = 1 + \mathbf{k} \operatorname{sn}^2(\pi\eta, \mathbf{k}), \quad J_Y = 1 - \mathbf{k} \operatorname{sn}^2(\pi\eta, \mathbf{k}), \quad J_Z = \operatorname{cn}(\pi\eta, \mathbf{k}) \operatorname{dn}(\pi\eta, \mathbf{k}), \quad \mathbf{k} = \frac{\theta_2^2(0|q)}{\theta_3^2(0|q)}.$$

Also, z denotes the spectral parameter which plays an important role in quantum integrable models. When $q \rightarrow 0$, due to $\mathbf{k} \rightarrow 0$ as well as

$$\operatorname{sn}(\pi\eta, 0) = \sin \pi\eta, \quad \operatorname{cn}(\pi\eta, 0) = \cos \pi\eta, \quad \operatorname{dn}(\pi\eta, 0) = 1,$$

one yields a XXZ chain with

$$J_X = J_Y = 1, \quad J_Z = \cos \pi\eta.$$

Remark that $2J_Z = \mathbf{q} + \mathbf{q}^{-1}$ where $\mathbf{q} = \exp(\pi i \eta)$ is referred to as the deformation parameter \mathbf{q} of the quantum group $U_{\mathbf{q}}(\mathfrak{sl}_2)$.

In fact, three R -matrices acting on $V_1 \otimes V_2 \otimes V_3$ satisfy the famous Yang-Baxter relation:

$$R_{12}(z) R_{13}(z + w) R_{23}(w) = R_{23}(w) R_{13}(z + w) R_{12}(z).$$

The subscript of, say, $R_{13}(z)$ means that it acts on $V_1 \otimes V_3$. From these R -matrices, one can construct the monodromy matrix $T_a(z)$ acting on $V_a \otimes (V_1 \otimes \cdots \otimes V_M)$:

$$T_a(z) = R_{aM}(z) \cdots R_{a1}(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

One can further yield the transfer matrix $\mathcal{T}(z) = \operatorname{tr}_a T_a(z)$ by performing a trace over the auxiliary space V_a . Utilizing the above Yang-Baxter relation repeatedly, one arrives at the so-called RTT relation:

$$R_{ab}(z - w) T_a(z) T_b(w) = T_b(w) T_a(z) R_{ab}(z - w),$$

from which the commutativity of transfer matrices follows:

$$[\mathcal{T}(z), \mathcal{T}(w)] = 0.$$

Let us briefly explain why there exists a common \mathbf{q} between the XXZ Hamiltonians H_{XXZ} and $U_{\mathbf{q}}(\mathfrak{sl}_2)$ encountered above. First, one can construct the XXZ transfer matrix from a product of R -matrices of the affine quantum group $U_{\mathbf{q}}(\widehat{\mathfrak{sl}}_2)$. Then, in order to derive H_{XXZ} the standard way is to take the logarithmic derivative of the XXZ transfer matrix.

2.1 Baxter's TQ relation as non-stationary Lamé equation

Baxter's Q -operator method is a powerful tool for finding the eigenvalue of transfer matrices. Let us briefly sketch his approach here. One prepares a local matrix $S_{aj}(z)$ which acts on $W_a \otimes V_j$ where $W_a = \mathbf{C}^L$ when $\exp(\pi i \eta L) = 1$. From $S_{aj}(z)$ we construct a global matrix

$$Q_a(z) = S_{aM}(z) \cdots S_{a1}(z)$$

acting on $W_a \otimes (V_1 \otimes \cdots \otimes V_M)$. Baxter's Q -operator is defined by $\mathcal{Q}(z) = \text{tr}_{W_a} Q_a(z)$ which acts also on the previous $V = V_1 \otimes \cdots \otimes V_M$. Baxter's idea was to consider the product of $\mathcal{T}(z)$ and $\mathcal{Q}(z)$

$$\begin{aligned} \mathcal{T}(z)\mathcal{Q}(z) &= \text{tr}_{V_a \otimes W_{a'}} \left\{ \prod_{j=1}^M R_{aj}(z) S_{a'j}(z) \right\} \\ &= \text{tr}_{V_a \otimes W_{a'}} \left\{ \prod_{j=1}^M U R_{aj}(z) S_{a'j}(z) U^{-1} \right\} \end{aligned}$$

followed by a gauge transformation: $R_{aj}(z) S_{a'j}(z) \rightarrow U R_{aj}(z) S_{a'j}(z) U^{-1}$ such that the latter becomes a triangular matrix via a suitable U . By doing so, both eigenvalues of $\mathcal{T}(z)$ and $\mathcal{Q}(z)$ are shown to satisfy Baxter's TQ relation [17]²

$$\mathcal{T}(z)\mathcal{Q}(z) = \phi(z - \frac{\eta}{2})\mathcal{Q}(z + \eta) + \phi(z + \frac{\eta}{2})\mathcal{Q}(z - \eta) \quad (2.2)$$

with $\phi(z) = \theta_1^M(z|q)$.

At the Razumov-Stroganov point³ $\eta = 1/3$ [15], a particularly simple expression for the ground-state eigenvalue of $\mathcal{T}(z)$ was conjectured to be $\phi(z)$ [14, 15, 24]. Their conjecture holds only when the number of chain sites is odd: $M = 2n + 1$ ($n \in \mathbf{Z}_{\geq 0}$). Inserting this $\mathcal{T}(z)$ into (2.2), Bazhanov and Mangazeev [16] managed to show that Q -operators dressed by

$$\Psi_{\pm}^{(8\text{vertex})}(z, q, n) = \frac{\theta_1^{2n+1}(z|q)}{\theta_1^n(3z|q^3)} \mathcal{Q}_{\pm}(z, q, n)$$

satisfy the non-stationary Lamé equation:

$$6q \frac{\partial}{\partial q} \Psi_{\pm}^{(8\text{vertex})}(z, q, n) = \frac{1}{\pi^2} \left\{ -\frac{\partial^2}{\partial z^2} + 9n(n+1)\wp(3z|q^3) + c(q, n) \right\} \Psi_{\pm}^{(8\text{vertex})}(z, q, n). \quad (2.3)$$

²See [18, 19, 20, 21, 22] for recent applications of Baxter's TQ relation to 4d gauge theories on Ω -backgrounds and Nekrasov's partition function [23].

³That η differs from the typical value $\pi/3$ is due to our choice of two half-periods $(\omega_1, \omega_2) = (1/2, \tau/2)$ of Weierstrass's elliptic function $\wp(z|q) \equiv \wp(z|\omega_1, \omega_2)$ instead of $(\pi/2, \pi\tau/2)$. These two notations are related by $\wp(tz|t\omega_1, t\omega_2) = t^{-2}\wp(z|\omega_1, \omega_2)$.

Let two half-periods of Weierstrass's elliptic function $\wp(z|q) \equiv \wp(z|\omega_1, \omega_2)$ be $(\omega_1, \omega_2) \equiv (1/2, \tau/2)$. Then,

$$\begin{aligned}\wp(z|q) &= -\zeta'(z|q), \\ \zeta(z|q) &= \frac{\theta_1'(z|q)}{\theta_1(z|q)} + 2\eta_1(q)z, \\ \eta_1(q) &= 4\pi^2 \left(\frac{1}{24} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right) = -\frac{1}{6} \frac{\theta_1'''(0|q)}{\theta_1'(0|q)}, \\ c(q, n) &= 18n(n+1)\eta_1(q^3).\end{aligned}$$

In terms of the new variable $s = 3z$, one can rewrite (2.3) into

$$\frac{2}{3}q \frac{\partial}{\partial q} \Psi_{\pm}^{(8\text{vertex})}(s/3, q, n) = \frac{1}{\pi^2} \left\{ -\frac{\partial^2}{\partial s^2} + n(n+1) \left(\wp(s|q^3) + 2\eta_1(q^3) \right) \right\} \Psi_{\pm}^{(8\text{vertex})}(s/3, q, n). \quad (2.4)$$

We can replace $2q\partial/3\partial q$ by $2\partial/\pi i \partial \bar{\tau}$ with $3\tau = \bar{\tau}$.

3. WZW model side

Our goal is to see the appearance of (2.3) within the context of WZW model and then interpret Stroganov's claim geometrically.

3.1 Affine Lie algebra

The conformal symmetry here will be realized by means of the level- k affine Lie algebra $\widehat{\mathfrak{g}}$. In general, the integrable irreducible $\widehat{\mathfrak{g}}$ -module $L_{k,\lambda}$ is characterized by a set of non-negative highest weights λ_a ($a = 0, \dots, r = \text{rank}$) w.r.t. \mathfrak{g} (simple finite-dimensional Lie algebra) where $\lambda_0 = k - (\theta, \lambda) \geq 0$. Symbolically,

$$\lambda_a \in P_+^k = \{\lambda_a \in P_+ \mid 0 \leq (\theta, \lambda) \leq k\}, \quad a = 1, \dots, r.$$

$L_{k,\lambda}$ with all null states being decoupled forms an unitary representation of $\widehat{\mathfrak{g}}$.⁴ Let us proceed to explain various notations used above.

We focus only on the A_{N-1} -type Lie algebra \mathfrak{sl}_N whose $N^2 - 1$ generators can get triangularly decomposed into $\mathfrak{sl}_N = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (\mathfrak{h} : Cartan subalgebra, $r = \dim \mathfrak{h}$). Consider its $(N^2 - 1)$ -dimensional adjoint representation labeled by a root system $\Delta = \Delta_+ \cup \Delta_-$,

⁴Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$. Given the highest weight state $|\lambda\rangle$ which is annihilated by generators in $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (\mathfrak{g} \otimes \mathbb{C}[z])$, the reducible module is gained by applying to $|\lambda\rangle$ repeatedly generators in $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus (\mathfrak{g} \otimes \mathbb{C}[z^{-1}])$. In order to decouple null states from the module, one must further impose

$$(E_{\theta} \otimes z^{-1})^{\lambda_0+1} |\lambda\rangle = 0, \quad (E_{-\alpha})^{(\alpha^\vee, \lambda)+1} |\lambda\rangle = 0, \quad \theta : \text{highest root}.$$

i.e. a set of vectors in a r -dimensional lattice. Given one positive (non-zero) root vector $\alpha \in \Delta_+$, we can choose $E_\alpha = E_{i,j} \in \mathfrak{n}_+$ ($i < j$). Note that $E_{i,j}$ is an $N \times N$ matrix with its (i, j) -th entry unity and zero otherwise. Take for example $\Phi = \text{diag}(x_1, \dots, x_N) \in \mathfrak{h}$ (subject to $x_1 + \dots + x_N = 0$).⁵ There holds

$$[\Phi, E_\alpha] = \alpha(\Phi)E_\alpha, \quad \alpha(\Phi) = x_i - x_j. \quad (3.1)$$

Because all roots are located in a r -dimensional lattice and only r of them are independent, let $\bar{\alpha}_a$ be simple roots such that $\alpha = \sum_{a=1}^r m_a \bar{\alpha}_a \in \Delta_+$ if $m_a \in \mathbf{Z}_{\geq 0}$. Generators in Cartan subalgebra are normalized by the length constraint $(\alpha, \alpha) = 2$ where the inner product is the usual one. In other words, given an orthonormal basis $\{e_1, \dots, e_N\}$ obeying $(e_i, e_j) = \delta_{ij}$ let $\alpha \equiv e_i - e_j \in \Delta$ ($1 \leq i \neq j \leq N$). For positive roots in Δ_+ , $i < j$. For simple roots, $j = i + 1$. Consequently, we are led to

$$[E_\alpha, E_{-\alpha}] = E_{i,i} - E_{j,j} \equiv H_\alpha, \quad [H_\alpha, E_\alpha] = \alpha(H_\alpha)E_\alpha = 2E_\alpha,$$

which implies that the weight of each root vector is just encoded in Δ .

From now on, we adopt the so-called Weyl-Cartan basis for \mathfrak{g} . That is, define $\mathbf{H} = (H_1, \dots, H_r) \in \mathfrak{h}$ such that the highest weight state $|\lambda\rangle$ satisfies $H_a|\lambda\rangle = \lambda_a|\lambda\rangle$. When it comes to roots, for $\alpha = (\alpha_1, \dots, \alpha_r) \in \Delta_+$ one has under this basis

$$[H_a, H_b] = 0, \quad [E_\alpha, E_{-\alpha}] = \sum_{a=1}^r \alpha_a^\vee H_a, \quad [H_a, E_\alpha] = \alpha_a E_\alpha. \quad (3.2)$$

Here, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ and again $(\alpha, \alpha) = 2$ is imposed as a normalization of \mathbf{H} where the inner product $(\alpha, \alpha) = \sum_a \alpha_a \alpha_a$.

Let $\bar{\alpha}^\vee$'s be simple coroots represented by

$$\bar{\alpha}_a^\vee = \sum_{b=1}^r A_{ab} \Lambda_b, \quad A_{ab} = (\bar{\alpha}_a, \bar{\alpha}_b^\vee) : \text{Cartan matrix.}$$

A set of fundamental weights $\{\Lambda_a\}$ is used to express the highest weight vector as $\lambda = \sum_a \lambda_a \Lambda_a$ with $(\bar{\alpha}_a^\vee, \Lambda_b) = \delta_{ab}$. The level k of $\widehat{\mathfrak{sl}}_N$ is given by

$$k = \lambda_0 + (\theta, \lambda) = \sum_{a=0}^r \lambda_a, \quad \theta = \sum_{a=1}^r \mathbf{a}_a^\vee \bar{\alpha}_a^\vee = \sum_{a=1}^r \bar{\alpha}_a^\vee, \quad \mathbf{a}_a^\vee : \text{colabel,}$$

where $\kappa = k + h^\vee$ ($h^\vee = \sum_{a=0}^r \mathbf{a}_a^\vee$: dual Coxeter number). For the A -type Lie algebra, $\mathbf{a}_0^\vee = \mathbf{a}_a^\vee = 1$. While the level k goes to infinity, the integrable irreducible $\widehat{\mathfrak{g}}$ -module reduces to that of \mathfrak{g} .

⁵This constraint will correspond to decoupling the center of motion associated with the non-stationary N -body Lamé equation.

3.2 KZB equation

We move to discuss correlation functions in WZW model. Of interest are their chiral parts, conformal blocks, satisfying Knizhnik-Zamolodchikov equations [6, 11]. However, it is necessary for us to first get familiar with constructing Virasoro algebra from affine Lie ones.

Let J_n^I ($n \in \mathbf{Z}$, $I, J = 1, \dots, \dim \mathfrak{g}$) be generators of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[z, z^{-1}] \oplus c\mathbf{C}$ whose central extension is $c = k\dim \mathfrak{g}/\kappa$ where

$$12\rho^2 = h^\vee \dim \mathfrak{g}, \quad \rho = \sum_{a=1}^r \Lambda_a = \sum_{\alpha \in \Delta_+} \frac{\alpha}{2} : \text{Weyl vector.}$$

According to Sugawara's construction, the generator of Virasoro algebra can be expressed via J_n^I :

$$2\kappa L_n = \sum_{m \in \mathbf{Z}} g_{IJ} : J_{n-m}^I J_m^J := g_{IJ} \left(\sum_{m < 0} J_m^I J_{n-m}^J + \sum_{m \geq 0} J_{n-m}^I J_m^J \right)$$

where g_{IJ} is the inverse of $g^{IJ} = K(\mathfrak{g}_I, \mathfrak{g}_J)$ called Killing form:

$$K(H_a, H_b) = \delta_{ab}, \quad K(E_\alpha, E_{-\alpha}) = \frac{2}{\alpha^2}, \quad K(\cdot, \cdot) = \text{zero otherwise.}$$

In particular, κL_0 is translated into the quadratic Casimir operator Ω of \mathfrak{g} :

$$\Omega = \frac{1}{2} \sum_{a=1}^r H_a^2 + \frac{1}{4} \sum_{\alpha \in \Delta_+} \alpha^2 (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha), \quad \alpha^2 = (\alpha, \alpha) \quad (3.3)$$

when applied to the highest weight state $|\lambda\rangle$. Following (3.2), we see that L_0 actually measures the conformal dimension Δ_λ of $|\lambda\rangle$ in the module $L_{k,\lambda}$:

$$\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa} \quad (3.4)$$

where the inner product is taken w.r.t. the preceding Weyl-Cartan basis.

Our main concern are KZB equations which look like

$$\kappa \frac{\partial}{2\pi i \partial \tau'} \Psi = \frac{1}{4} H_0 \Psi, \quad H_0 = -\frac{1}{2\pi^2} \sum_{a=1}^r \partial_{u_a} \partial_{u_a} + 2 \sum_{\alpha \in \Delta} p(e^{\alpha(U)}) E_\alpha E_{-\alpha}, \quad (3.5)$$

where

$$U = 2\pi i \sum_{a=1}^r u_a H_a, \quad p(t) = - \sum_{m \in \mathbf{Z}} \frac{q'^m t}{(1 - q'^m t)^2}, \quad q' = e^{2\pi i \tau'} \quad (3.6)$$

and $\alpha(U)$ is defined in (3.1). Certainly, (3.5) is derived by applying the Ward-Takahashi identity associated with the energy-momentum tensor $T(z) = \sum_{n \in \mathbf{Z}} z^{-n-2} L_n$ to one-point toric conformal blocks Ψ :

$$\Psi = \text{Tr}_{L_{k,\lambda}} \left(q'^{L_0 - \frac{c}{24}} e^U v_\ell(z) \right), \quad \ell \in P_+^k, \quad c = \frac{k\dim \mathfrak{g}}{\kappa}. \quad (3.7)$$

Notice that for \mathfrak{sl}_2 the primary field $v_\ell(z)$ is V -valued (taking its value in V) and acted on by ρ_ℓ given certain spin- $\ell/2$ \mathfrak{sl}_2 representation (ρ_ℓ, V_ℓ) . Still, the marked point z located on the torus (complex moduli τ') can be sent to zero because Ψ satisfies

$$\left(z \frac{\partial}{\partial z} + \rho_\ell(\kappa^{-1}\Omega)\right)\Psi = 0. \quad (3.8)$$

By definition, without any $v_\ell(z)$ inserted Ψ reduces to the affine character associated with the integrable module $L_{k,\lambda}$:

$$\Psi \rightarrow \chi = \text{Tr}_{L_{k,\lambda}} \left(q^{L_0 - \frac{c}{24}} e^U \right). \quad (3.9)$$

Let us pause for a while to discuss the V -valuedness of $v_\ell(z)$. This can be done twofold. First, define $v(z) \equiv v(\zeta|z)$ which depends additionally on an internal coordinate ζ . Its OPE with some $\widehat{\mathfrak{sl}}_2$ -current field $J^I(w)$ reads

$$J^I(w)v(\zeta|z) \sim \frac{1}{w-z} \mathcal{D}^I(\zeta)v(\zeta|z), \quad \mathcal{D}^I(\zeta)v(\zeta|z) \equiv \rho(J^I)v(z).$$

Second, we resort to Wakimoto's representation. Introduce the primary field corresponding to the $\widehat{\mathfrak{sl}}_2$ highest weight state $|\ell\rangle$ in terms of a chiral free boson $\varphi(z)$:

$$|\ell\rangle \rightarrow : \exp(\ell/\sqrt{2})\varphi(z) :$$

whose conformal dimension Δ_ℓ is just computed in (3.4) with $\lambda \rightarrow \ell$. Instead of the additional ζ -dependence, one prepares another chiral free field $\gamma(z)$ ⁶ and constructs the full \mathfrak{sl}_2 spin- $\ell/2$ multiplet which contains

$$\gamma(z)^{\ell/2-m} : \exp(\ell/\sqrt{2})\varphi(z) :, \quad m = -\ell/2, \dots, \ell/2. \quad (3.10)$$

Then, $v_\ell(z)$ can be identified with one of them.

From (3.7) we realize that the role of $v_\ell(z)$ is an intertwiner, i.e. $v_\ell(z) : L_{k,\lambda} \rightarrow L_{k,\lambda} \otimes V[0]$. Here, $V[0]$ stands for the one-dimensional zero-weight subspace of V_ℓ , $(\ell+1)$ -dimensional \mathfrak{sl}_2 -module.⁷ Due to the Ward-Takahashi identity w.r.t. $\widehat{\mathfrak{sl}}_2$ -current fields applied to Ψ , we see (H : \mathfrak{sl}_2 Cartan generator)

$$\rho_\ell(H)\Psi = 0. \quad (3.11)$$

This explains why $v_\ell(z)$ belongs to the zero-weight subspace $V[0] \subset V_\ell$.

3.3 H_0

Let us describe H_0 in (3.5) in more detail. We want to look into the function $p(t)$ [7, 9, 25, 26] inside H_0 . By using Weierstrass's \wp -function it gets expressed by ($t = e^{2\pi i w}$)

$$\begin{aligned} p(t) &= -\frac{t}{(1-t)^2} - \sum_{m \neq 0} \frac{q^m t}{(1-q^m t)^2}, \\ 4\pi^2 p(t) &= -\partial_w^2 \log \theta_1(w|q'^{\frac{1}{2}}) = \wp(w|q'^{\frac{1}{2}}) + 2\eta_1(q'^{\frac{1}{2}}). \end{aligned} \quad (3.12)$$

⁶Note that $\mathbf{dim}(\Delta_+)$ is equal to the total number of pairs of $(\beta(z), \gamma(z))$.

⁷ $V[0]$ vanishes if $\mathbf{dim}(V) = \ell + 1$ ($\ell = 0, 1, 2, \dots$) is even.

Here, \wp , θ_1 and η_1 follow the same convention adopted in section 2. Eq. (3.12) is explained as below. Because $p(t)$ has order-two poles at $\{q'^m\}_{m \in \mathbf{Z}}$ and satisfies the periodicity condition $p(q't) = p(t)$, it can be rewritten into the form

$$p(t) = -\frac{t}{(1-t)^2} - \sum_{m>0} \left\{ \frac{q'^m t}{(1-q'^m t)^2} + \frac{q'^m t^{-1}}{(1-q'^m t^{-1})^2} \right\}.$$

Recalling $t = e^{2\pi i w}$ we find the first term becomes

$$-\frac{t}{(1-t)^2} = \frac{1}{4 \sin^2(\pi w)},$$

and other terms become

$$-\frac{q'^m t}{(1-q'^m t)^2} - \frac{q'^m t^{-1}}{(1-q'^m t^{-1})^2} = -2q'^m \frac{\cos(2\pi w)(1+q'^{2m}) - 2q'^m}{(1-2q'^m \cos(2\pi w) + q'^{2m})^2}.$$

On the other hand, based on the product representation of $\theta_1(w|q'^{\frac{1}{2}})$:

$$\theta_1(w|q'^{\frac{1}{2}}) = 2q'^{\frac{1}{8}} \sin(\pi w) \prod_{m>0} (1-q'^m)(1-2q'^m \cos(2\pi w) + q'^{2m})$$

we have

$$-\partial_w^2 \log \theta_1(w|q'^{\frac{1}{2}}) = \frac{\pi^2}{\sin^2(\pi w)} - \sum_{m>0} 8\pi^2 q'^m \frac{\cos(2\pi w)(1+q'^{2m}) - 2q'^m}{(1-2q'^m \cos(2\pi w) + q'^{2m})^2}.$$

Combined with

$$-\partial_w^2 \log \theta_1(w|q'^{\frac{1}{2}}) = \wp(w|q'^{\frac{1}{2}}) + 2\eta_1(q'^{\frac{1}{2}}),$$

we go back to (3.12). To explicitly evaluate $e^{\alpha(U)}$ for $p(t)$, we resort to (3.1). Generally, in the case of \mathfrak{sl}_N

$$\frac{1}{2\pi i} U = \sum_{a=1}^r u_a H_a = \text{diag}(y_1, \dots, y_N), \quad \sum_{i=1}^N y_i = 0.$$

Due to $(\alpha, \alpha) = 2$ as stressed, one sees $\alpha(U) = 2\pi i \sqrt{2} u_1$ and $w \rightarrow \sqrt{2} u_1 \equiv u$ for \mathfrak{sl}_2 .

Finally, we want to determine the eigenvalue of $E_\alpha E_{-\alpha}$ in (3.5) which acts on $V[0] \subset V_\ell$ of the primary field $v_\ell(z)$. Since $V[0] = \mathbf{C}$ is one-dimensional, in view of (3.10) one can assume that it is spanned by some monomial like $(L_1 \cdots L_N)^\xi$. Furthermore, for \mathfrak{sl}_N there exists the following representation:

$$E_\alpha \equiv E_{ij} = L_i \frac{\partial}{\partial L_j}, \quad i \neq j = 1, \dots, N, \quad \alpha \in \Delta$$

whereas

$$H_i = L_i \frac{\partial}{\partial L_i} - L_{i+1} \frac{\partial}{\partial L_{i+1}}, \quad i = 1, \dots, N-1.$$

In the case of $N = 2$ (or \mathfrak{sl}_2) we thus obtain the eigenvalue $\ell(\ell+2)/4$ of $E_\alpha E_{-\alpha}$ through $\xi \equiv \ell/2$. This choice of ξ is rigid and not arbitrary.

3.4 Comparison

Equipped with these, we are in a position to replace H_0 in (3.5) by

$$H_0 = \frac{1}{\pi^2} \left\{ -\frac{\partial^2}{\partial u^2} + \frac{\ell(\ell+2)}{4} \left(\wp(u|q'^{\frac{1}{2}}) + 2\eta_1(q'^{\frac{1}{2}}) \right) \right\}, \quad u \equiv \sqrt{2}u_1.$$

We then arrive at the familiar form of KZB equations:

$$\kappa \frac{\partial}{2\pi i \partial \tau'} \Psi = \frac{1}{4\pi^2} \left\{ -\frac{\partial^2}{\partial u^2} + \frac{\ell(\ell+2)}{4} \left(\wp(u|q'^{\frac{1}{2}}) + 2\eta_1(q'^{\frac{1}{2}}) \right) \right\} \Psi. \quad (3.13)$$

Let us slightly rewrite (3.13) into

$$2\pi\kappa \frac{\partial}{i\partial \tau'} \tilde{\Psi} = \left\{ -\frac{\partial^2}{\partial u^2} + \frac{\ell(\ell+2)}{4} \wp(u|q'^{\frac{1}{2}}) \right\} \tilde{\Psi} \quad (3.14)$$

with

$$\tilde{\Psi} = \exp \left(-\frac{i}{2\pi\kappa} \frac{\ell(\ell+2)}{4} \int^{\tau'} 2\eta_1 d\tau'' \right) \Psi. \quad (3.15)$$

Remark that by $(\tau', u) \rightarrow \kappa^{-1}(\tau', u)$ and $\tilde{\Psi} \rightarrow \Phi$ (see footnote 3) (3.14) becomes

$$2\pi \frac{\partial}{i\partial \tau'} \Phi = \left\{ -\frac{\partial^2}{\partial u^2} + \frac{\ell(\ell+2)}{4} \wp(u|q'^{\frac{1}{2}}) \right\} \Phi. \quad (3.16)$$

Exactly the same procedure of redefining the wave function $\Psi_{\pm}^{(8\text{vertex})}$ by absorbing η_1 into it can be applied to (2.4). After doing that, comparing (2.4) with (3.13) we find

$$s \Longleftrightarrow u, \quad n \Longleftrightarrow \ell/2, \quad \bar{\tau} \Longleftrightarrow \tau'. \quad (3.17)$$

The relationship (3.17) reveals that keeping the total spin-chain site number M odd [14] can now be interpreted as the existence requirement for one-point toric conformal blocks due to $M = 2n + 1 = \mathbf{dim} V$ in view of (3.11). This serves as a geometric interpretation of Stroganov's claim. We will provide another consistency check in section 4.

4. Test: reduction to Sutherland model

Performing a trigonometric reduction to the spin-1/2 XXZ chain ($q \rightarrow 0$) helps strengthen the correspondence indicated in (3.17). On WZW model side, this leads to a degenerate torus drawn in Fig. 2.⁸

Based on $c(0, n) = 3n(n+1)$ and

$$\Psi_{\pm}^{(8\text{vertex})}(s/3, q, n) = q^{\frac{3}{2}(d_{\pm} + \frac{1}{4})} \Psi_{\pm}^{(6\text{vertex})}(s/3, n) (1 + \mathcal{O}(q)), \quad d_{\pm} = \frac{1 \mp 6}{36},$$

⁸See also [27, 28] where the issue presented here was encountered within the context of 2d Liouville CFT/4d $\mathcal{N} = 2$ gauge theory correspondence initiated in [29, 30, 31, 32].

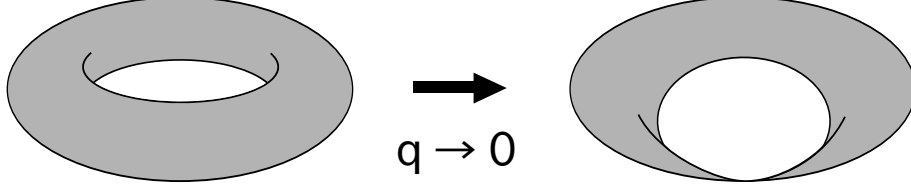


Figure 2: As $q \rightarrow 0$ one has a degenerate torus.

one finds that eq. (2.4) becomes

$$\left\{ -\frac{\partial^2}{\partial s^2} + n(n+1) \left(\wp(s|0) + \frac{1}{3} \right) - \pi^2 \left(d_{\pm} + \frac{1}{4} \right) \right\} \Psi_{\pm}^{(6\text{vertex})}(s/3, n) = 0.$$

Furthermore, from

$$\wp(s|0) = \frac{\pi^2}{\sin^2 \pi s} - \frac{1}{3}, \quad \frac{\theta'_1(s|q)}{\theta_1(s|q)} \xrightarrow{q \rightarrow 0} \pi \frac{\cos \pi s}{\sin \pi s},$$

we arrive at the two-body Sutherland model:

$$\left\{ -\frac{\partial^2}{\partial s^2} + \frac{\pi^2 n(n+1)}{\sin^2 \pi s} - \pi^2 \left(d_{\pm} + \frac{1}{4} \right) \right\} \Psi_{\pm}^{(6\text{vertex})}(s/3, n) = 0. \quad (4.1)$$

Then, through⁹

$$\tilde{\Psi}_{\pm}^{(6\text{vertex})}(s) = (\sin \pi s)^{-n-1} \Psi_{\pm}^{(6\text{vertex})}(s/3, n)$$

we are able to rewrite (4.1) into ($\pi s = \tilde{s}$)

$$\left\{ -\frac{\partial^2}{\partial \tilde{s}^2} - 2(n+1) \cot \tilde{s} \frac{\partial}{\partial \tilde{s}} + (n+1)^2 - \left(d_{\pm} + \frac{1}{4} \right) \right\} \tilde{\Psi}_{\pm}^{(6\text{vertex})}(s) = 0.$$

In fact, $\tilde{\Psi}_{\pm}^{(6\text{vertex})}(s)$ is related to the Gegenbauer polynomial $G_e^{(\nu)}(\cos \tilde{s})$:

$$G_e^{(\nu)}(\cos \tilde{s}) = \frac{\Gamma(e+2\nu)}{\Gamma(2\nu)e!} {}_2F_1 \left(-e, e+2\nu, \nu + \frac{1}{2}; \frac{1 - \cos \tilde{s}}{2} \right)$$

through $\nu = n+1$ and $e(e+2\nu) = (d_{\pm} + \frac{1}{4}) - (n+1)^2$.

On the other hand, it is also known that $\tilde{\Psi}_{\pm}^{(6\text{vertex})}(s)$ by changing the variable to $S = \exp(i\tilde{s})$ becomes the Jack polynomial $J_{\lambda}^{(\nu)}(S)$ where λ denotes the energy level. More precisely, by $\Delta = (\sin \tilde{s})^{n+1}$ the Hamiltonian H_S of the two-body Sutherland model is transformed into

$$H_0 = \Delta^{-1}(H_S - e_0)\Delta, \quad H_S = -\frac{\partial^2}{\partial \tilde{s}^2} + \frac{n(n+1)}{\sin^2 \tilde{s}},$$

⁹Another transformation:

$$\tilde{\Psi}_{\pm}^{(6\text{vertex})}(s) = (\sin \pi s)^n \Psi_{\pm}^{(6\text{vertex})}(s/3, n)$$

will lead to Stroganov's result [14].

whose eigenfunction is the Jack polynomial $J_\lambda^{(\nu)}(S)$. Here, e_0 stands for the eigenvalue of H_0 w.r.t. the ground-state Δ . In addition, as $n \rightarrow 0$ $J_\lambda^{(\nu)}(S)$ reduces to the Schur function:

$$\chi_\lambda(S) = \frac{S^{1+\lambda} - S^{-1-\lambda}}{S - S^{-1}}. \quad (4.2)$$

Eq. (4.2) has its \mathfrak{sl}_N analogy, i.e. given $\mathbf{S} = \text{diag}(S_1, \dots, S_N)$ ($\det \mathbf{S} = 1$) one has

$$\chi_R(\mathbf{S}) = \frac{\det(S_i^{R_j+N-j})}{\det(S_i^{N-j})}, \quad i, j = 1, \dots, N,$$

where $R = (R_1, \dots, R_{N-1}, 0)$ stands for a Young tableau. Each row length of R obeys $R_1 \geq R_2 \geq \dots$. In addition, there exists

$$(R_1 - R_2, \dots, R_i - R_{i+1}, \dots, R_{N-1}) = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{N-1})$$

between R and $\lambda \in P_+$ of \mathfrak{sl}_N .

To see the emergence of (4.2) on CFT side, we follow two steps below whose order differs from the above procedure.

Step 1: As mentioned in section 3, when the insertion becomes an identity operator ($\ell \rightarrow 0$) the toric conformal block reduces to the level- k $\widehat{\mathfrak{sl}}_2$ character which is explicitly ($k \geq \lambda$, λ : highest weight)

$$\chi = \frac{\theta_{\sqrt{2}\lambda+1, k+2} - \theta_{-\sqrt{2}\lambda-1, k+2}}{\theta_{1,2} - \theta_{-1,2}},$$

$$\theta_{A,B} \equiv \theta_{A,B}(u_1|q') = \sum_{n \in \mathbf{Z} + \frac{A}{2B}} (q')^{Bn^2} e^{2Bn\pi i u_1}.$$

Step 2: Next, we take $\tau' \rightarrow i\infty$ such that only those terms involving $\mathbf{Z} = 0$ inside the summation of $\theta_{A,B}$ survive. The above affine character χ factorizes into two parts depending on respectively q' and u_1 :

$$\chi \xrightarrow{q' \rightarrow 0} q'^{\left(\frac{(\sqrt{2}\lambda+1)^2}{4\kappa} - \frac{1}{8}\right)} \frac{e^{\pi i(\sqrt{2}\lambda+1)u_1} - e^{-\pi i(\sqrt{2}\lambda+1)u_1}}{e^{\pi i u_1} - e^{-\pi i u_1}}.$$

Actually, the factor $\frac{(\sqrt{2}\lambda+1)^2}{4\kappa} - \frac{1}{8}$ comes from $L_0 - \frac{c}{24}$. Because when the degeneration depicted in Fig. 2 occurs only the highest weight state in the integrable module $L_{k,\lambda}$ contributes to χ . After dropping the q' -dependent part we are simply left with ($\rho = 1/\sqrt{2}$: Weyl vector)

$$\Psi \xrightarrow{\ell \rightarrow 0} \chi \xrightarrow{q' \rightarrow 0} \frac{e^{\pi i(\sqrt{2}\lambda+1)u_1} - e^{-\pi i(\sqrt{2}\lambda+1)u_1}}{e^{\pi i u_1} - e^{-\pi i u_1}} = \frac{e^{\pi i(\lambda+\rho)u} - e^{-\pi i(\lambda+\rho)u}}{e^{\pi i \rho u} - e^{-\pi i \rho u}}.$$

This process we are studying then serves as a confirmation of the link (Table 1) between WZW model and the spin-1/2 XYZ chain.

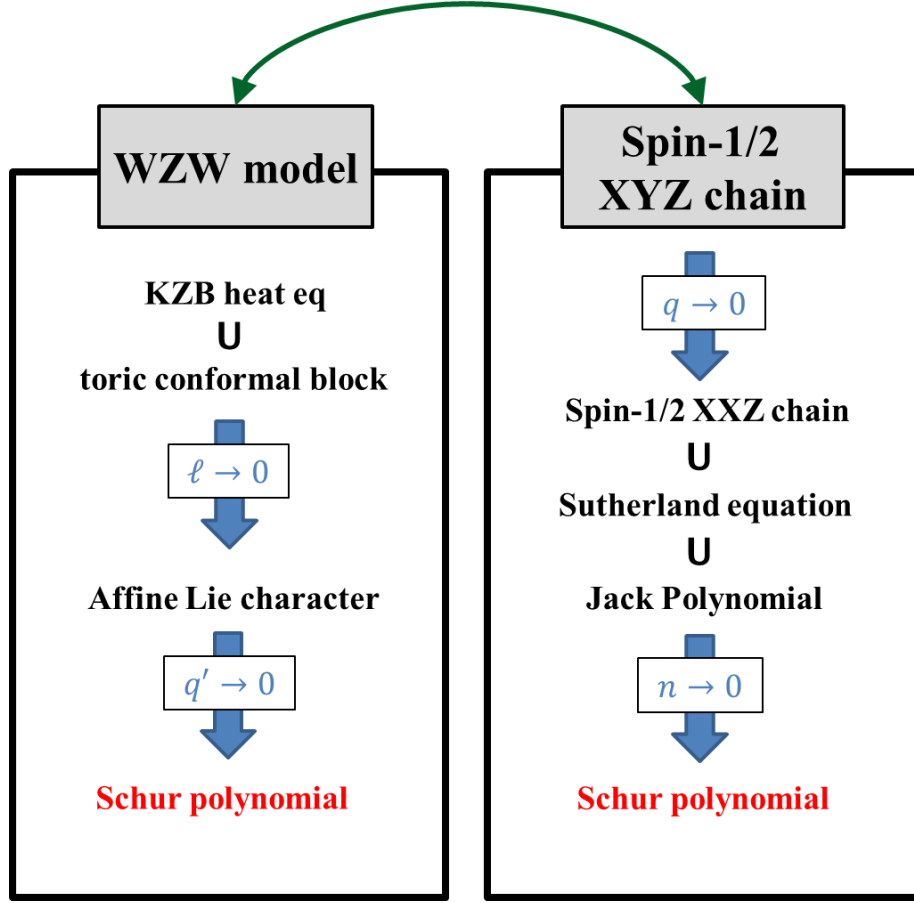


Figure 3: (RHS) A trigonometric degeneration of the spin-1/2 XYZ chain. (LHS) Through the identification $\ell/2 = n$, toric conformal blocks reduce correspondingly to Schur polynomials (eigenstates of the two-body Sutherland model in RHS as $n \rightarrow 0$). Note that the order of two limits differs between RHS and LHS.

5. Summary

We have provided an interpretation of Stroganov’s “The Importance of being Odd” at the Razumov-Stroganov point by means of CFT language. We found that the total number of the XYZ chain sites $M = 2n + 1$ is equal to the dimension of the \mathfrak{sl}_2 representation space V w.r.t. the primary field inserted on a torus in WZW model. Notice that M must be odd. The approach summarized in Fig. 3 was used to support our proposal.

In fact, there is still another interesting limit mentioned in [16], i.e. $n \rightarrow \infty$ and $q \rightarrow 0$ with $t = 8q^{\frac{3}{2}}n$ kept fixed as presented in Appendix. To study the corresponding deformation of WZW conformal blocks is an interesting future work, though the explicit form of them is not available. In the context of 2d Liouville field theory characterized by Virasoro algebra, similar issues have been addressed in [33, 34].

Acknowledgments

We thank professor Hratchya M. Babujian for helpful comments.

Appendix

Under $n \rightarrow \infty$ and $q \rightarrow 0$ with $t = 8q^{3/2}n$ kept fixed, one has $\Psi_{\pm}^{(8\text{vertex})} \rightarrow \mathcal{Q}_{\pm}(\theta, t)$.

By further adopting the new variable θ defined by $i\theta = s - \frac{1}{2}\pi\bar{\tau}$, this limit applied to (2.3) leads to a massive sine-Gordon model on a cylinder [16]:

$$t \frac{\partial}{\partial t} \mathcal{Q}_{\pm}(\theta, t) = \left\{ \frac{\partial^2}{\partial \theta^2} - \frac{1}{8} t^2 (\cosh 2\theta - 1) \right\} \mathcal{Q}_{\pm}(\theta, t).$$

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